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# NUMERICAL PROBLEMS IN THE CALCULATION OF ACOUSTIC SCATTERING WITH THE SOURCE SIMULATION TECHNIQUE

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*Abstract: In this work the source simulation technique was used to calculate the scattering of a plane wave by a circular cylinder with radial or elliptical transverse section. The extent to which the simulated field reproduces the original field depends on the degree of correspondence between the simulated and the given boundary conditions. Numerical simulations have shown that: 1) the shape of the auxiliary surface, 2) the number of sources, and 3) the way the sources are distributed are the most relevant parameters to ensure an accurate solution for the problem. In the case of the single-layer method, sources, should not be positioned close to the surface or to the center of the body, because the problem becomes ill-conditioned. The auxiliary surface and the scatterer should be as similar as possible in order to minimize the boundary error. With respect to the number of sources (N), there are two opposite effects: 1) if (N) is too small, the sound filed is not reproduced accurately; 2) if (N) is too lage, comuting timeincreaes and solution accuracy decreases. The method breaks down when excitation frequency coincides with the eigenfrequencies of the space formed by the auxiliary surface.*

*Keywords: scattering, eigenfrequencies, auxiliary surface.*

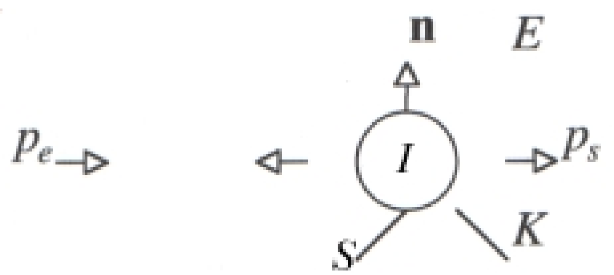
## 1. INTRODUCTION

The mathematical treatment of radiation and acoustic scattering represents a very old and much studied problem by mathematical physics (Lord Rayleigh, 1945; P.M. Morse, 1948). Radiation and scattering are present in all ondulatory phenomena (elastic waves in rigid bodies, electromagnetic waves, surface waves on the water, etc). The present study, however, deals only with “pure” acoustical waves, that is, acoustical waves in gases or liquids. Another important limitation is that all steps of the solution of the problem are considered linear. Consequently, the superposition principle is valid.

## 2. DESCRIPTION OF THE SCATTERING PROBLEM AND THE SOURCE SIMULATION TECHNIQUE

Considering a harmonic wave with sound pressure amplite $p_e$ in an infinite and homogeneous space $E$ which encounters in its displacement the body $K$, internal space is defined by $I$ , the scttered wave by $p_s$ and the surface of the body by $S$ . Over the surface of the body the unitary vector $\vec{n}$ is defined (see fig. 1).

Pressure and velicity of the particle can be determined as the result of the sum of the components $p_e$ and $p_s$ . Respectively




$$p_t = p_e + p_s \quad and \quad v_t = v_e + v_s \tag{2.1}$$

**Figure 1: Geometry of the acoustic scattering problem**

The complex sound pressure $p_t$ has to satisfy the Helmholtz equation

$$\Delta p_t + k^2 p_t = 0 \tag{2.2}$$

in $E$ , where $k = \omega / c$ is the wave number, $\omega$ is the circular frequency, $c$ the speed of sound. and $\Delta$ is the Laplace operator. Since sound scattering into the three-dimensional space is considered, the pressure $p_t$ has to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty}\left[\frac{\partial p_t}{\partial n} + jkp_t\right] = 0 \tag{2.3}$$

wich can be interpreted as a boundary condition at infinity. Here, $r = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ denotes the distance from $x$ to the origin, where we represent points in space by simple letters like $x = (x_1 + x_2 + x_3)$ . Solutions of the Helmholtz equation in $E$ wich also satisfy the radiation equation are called wave functions. To get a complete description of the problem, boundary conditions on the surface are needed. For the sacke of simplicity we only consider the so-called Neumann boundary value problem where the normal velocity and therefore the gradient of pressure

$$\frac{\partial p}{\partial n} = j\omega\rho v \tag{2.4}$$

is prescribe. Here $\rho$ is the density and $\partial / \partial n$ is the derivative in the direction of the outward normal.

The basic idea of the source simulation technique consists in replacing the scatterer body by a system of sources placed in the interior of the body. The sources are denoted by $\phi(x, y)$ , where $x$ is one arbritary point in space and $y$ is the position of the source singularity, i.e the location of the source point. Now, due to the source simulation technique in its most general form, it can be assumed that pressure can be represented by

$$p(x) = \iiint_Q A(y)\phi(x,y)dy \tag{2.5}$$

where $Q$ is a region which is fully contained in $I$ and embodies all sources. $A(y)$ is the yet unknown source density which gives every source a certain strenght. Every sigle function $\phi(x,y)$ can consist of a finite or infinite sum of elementary sources like e.g. monopoles, dipoles, etc. The volum integral in "Eq.(2.5)" reduces to a surface integral if the region $Q$ is a surface. Using line sources, a contour integral is obtained, and finally the integral turns into a finite sum if isolated point sources are used. The system functions $(\phi(x,y)|y \in Q)$ will be called the source system. All functions of the source system satisfy "Eq.(2.2)" by definition. If the source system also satisfies the boundary condition "Eq.(2.4)" on the surface $S$, then the sound field generated by the scatterer body and the field produced by the source are identical. This follows from the unique solvability of the exterior problem descrebed by "Eq.(2.2) and (2.3)" together with the local boundary conditions "Eq.(2.4)" (D. Colton an R. Kress, 1983). Hence such a source system will be called an equivalent source system. Consequently, the exact solution of the scattering (or radiation) problem can be found if it is possible to construct na equivalente source system. Solutions of the Helmholtz equation together with the radiation condition, especially in standar coordinate systems like spherical coordinates, are well-known. The problem is to satisfy the boundary conditions since the surface $S$ normally has a complicated geometry for pratically relevant problems, and analytical solutions are not available. In these cases only na approximate solution can be obtained, which means that the source system will satisfy the boundary conditions in an aproximate sense.

## 3. INFLUENCE PARAMETERS

The parameters that influence the performance of the method, that is, the capacity of the method to reproduce boundary conditions are the following: the type and number of sources, their posiotioning in the interior of the body, the shape of the source surface and the existence of critical frequencies.

### 3.1 Type of sources

By definition the sources must be radiating wave functions. It is convenient to work with available analytical functions. Only solutions solutions of the Helmholtz equation in separable coordinate systems can be constructed explicitly (P.M. Morse and H. Feshbach, 1953). In practice, spherical radiators (for three-dimensional problems) or cylindrical radiators (for two dimensional problems). A few attmpts to work with spheroidal functions can be found in the literature (R.H. Hackmann, 1984).

### 3.2 Location of the sources

A general assumption of the source simulation technique is that the sources must be located in the interior of the closed surface $S$. It is also possible to put sources on the boundary itself. But this leads to boundary integral equations and to the corresponding BEM, which are not topics of this paper. For the choice of the source location, essentially two alternatives are possible: 1) only a few source locations are chosen, but at these locations a source with increasing order is used; or 2) a continuous source distribution of simple sources on na inner auxiliary surface is employed. The constrast between boht method is the greatest if a closed auxiliary surface with a layer of monopoles is chosen as one extreme and a infinite series of multipoles at only one source location as the other extreme. The first method is called “the single-layer method” and the second “the one-point multipole method”. If the geometry of the body is spherical or cylindrical (or not far from those), the use of the one-point multipole is recommended with the multipol located in the center of the body. This procedure facilitates the convergence of the wave functions and reduces computing time. On the other hand, sources positioned very close to the center when using the single-layer method tend to cause thte matrix of linerarequations to became more ill-conditioned, leading to an increseased surface error (see fig. 2 and fig. 3). If the sources are positioned very close to the boundary, the accuracy will deteriorate due to the inadequate integration of the source singularity. There is again substancial increase in computing time due to the increase in the number of sources necessary in order to minimize surface error. Our findings are not in agreement wiht Bobrovnitzkii and Tomilina (1990), who say that the source surface should be close to the body surface in order to improve the accuracy of the problem reducing the boundary error. This is only correct when $kR$ is very large.

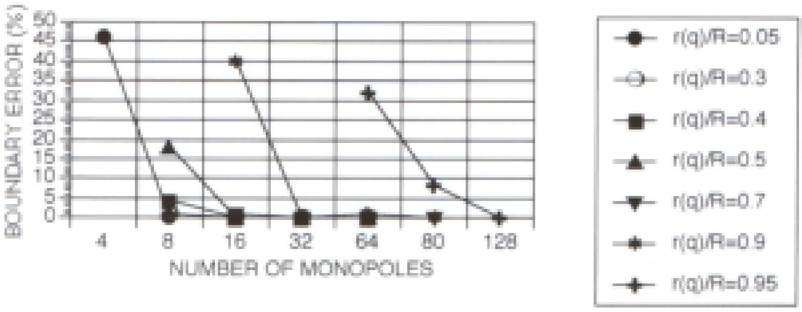



**Figure 2: Error when satisfying the boundary error as function of the position of the source surface with radius $r(q)$ and a circular cylinder with radius $R, kR = 0.73$.**

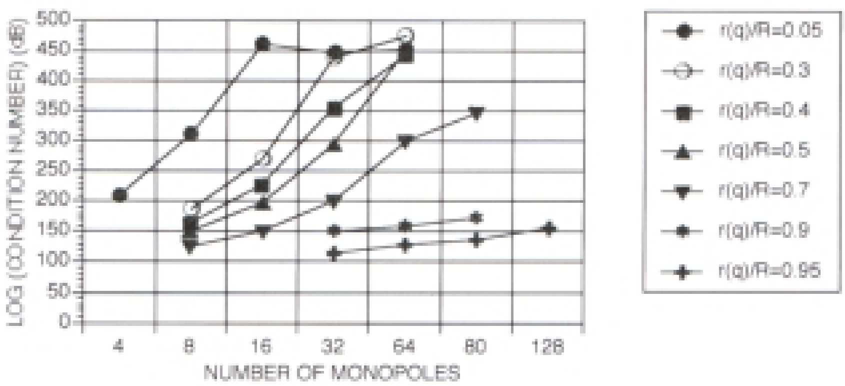



**Figure 3: Condition number in [dB] as a function of the position of the source surface with radius $r(q)$ and a circular cylinder of $R, kR = 0.73$.**

### 3.3 Number of sources

The number of sources is influenced by several parameters, but mainly by the geometry of the body an the type of the source. In the determination of the number of sources, with respect to the single-layer method considering a circular cylinder, an expression was intended which would give the smallest number of monopoles necessary to satisfy the boundary conditions. Additionally, the position $r(q)/R$ (see fig. 3) of the source surface should also comply with the assumption of a minimal number of monopoles. The simulation leads us to the following

$$N = \beta . kR \tag{2.6}$$

where $\beta$ is anknown factor, and $R$ is the cylinder radius. This expression establishes a relationship between the wave number, the size of the body, and the number of mononopoles. With the simulation the following values have been found for:

a) for $0.73 \leq kR \leq 1.46$ *and* $r(q)/R = 0.4 \rightarrow \beta = 16$
b) for $1.83 \leq kR \leq 3.66$ *and* $r(q)/R = 0.5 \rightarrow \beta = 8$
c) for $4.58 \leq kR \leq 7.33$ *and* $r(q)/R = 0.6 \rightarrow \beta = 6$
d) for $9.16 \leq kR \leq 23.10$ *and* $0.7 \leq r(q)/R \leq 0.8 \rightarrow \beta = 4$

**3.4 Shape of the source surface**

One aspect rarely conseidered in the utilization of the source simulation techniaque is the shape of the source, that is,the shape of hte auxiliary surface over which the sources are positioned (Zannin, 1996). The object of study here is a circular cylinder, and for the source surface the following shapes have benn used: cylindrical and elyptical. Surface error could be minimized and boundary condition satisfied in all tested cases. The number of monopole sources needed grew in direct proportion to the deviation of the source surface from a circular cylindrical shape.

A cylinder with elyptical transverse section was also used a second object of study. The results obtained were very consistent as long as the source surface was identical with the external surface (see fig. 4). For the case of a cylindrical source surface located within the elyptical body, boundary condition could not be satisfied (see fig. 5). The elyptical body required a significantly larger number of monopole sources in order to get boundary error minimized.

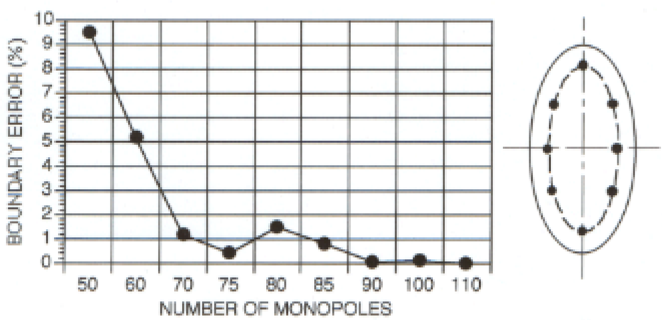



**Figure 4: Ellyptical cylinder with ellyptical source surface.**

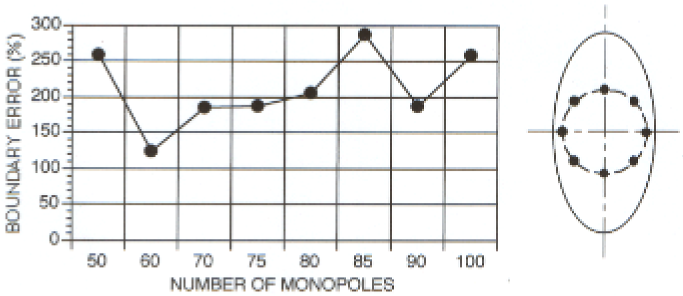



**Figure 5: Ellyptical cylinder with cylindrical source surface**

### 3.5 Critical Frequencies

There are frequencies, sometimes called fictitious eigenfrequencies at wich or close to which the solution of the Helmholtz integral equation is non-unique. It is well known that the boundary element method breaks down at these frequencies (Ochmann, 1990).

Jeans and Mathews (1992) have demonstrated that in the use of the source simulation technique the critical frequencies corresponde to the eigenfrequencies of the internal space formed by the closed source surface, when over this surface the boundary condition of Dirichlet is considered.

The eigenfrequencies of a circular space with the Dirichlet condition over its surface are represented by the roots of the Bessel function (E. Skudrzyk, 1971):

$$J_n(kr) = 0 \qquad n = 0,1,2,3... \tag{2.7}$$

where $r$ is the radius of the source surface.

Nevertheless, other authors report that they have not observed the presence of inner eigenfrequencies when utilizing the source simulation technique, both in the calculation of the acoustic radiation and of the aocustic scattering. Part of the conclusions ot these studies are cited in what follows: “The SUP solution does not appear to be affected by spurius internal resonances which have plagued the integral formulations in the past (P.S. Kondapelli et. all, 1991)”; “The results for $R = 1$ and $k = 2.40483$ which is the smallest zero of the Dirichlet eigenvalue for the circle, illustrate that there is no unique problem at the critical wave number (R. Kress and A. Mohsen, 1986)”.

In this work we tried to identify the presence or absence of the eigenfrequencies. Figure 6 shows the error when satisfying the boundary conditions close to and at the first frequency of resonance of a circle: $kr = 2.4048255577$. It is fairly obvious that there is a huge error at this frequency and that the problem formulation breaks down. In fig. 7 we have the logarithm of the condition number. One important conclusion that can be drawn from figures 6 and 7 is that resonance belongs to a very narrow range of frequencies. For source surfaces like a cylinder or a sphere resonance can be easily calculated and therefore avoided. This is one of the advantages of the source simulation technique with respect to the boundary element method. One question always present with respect to the source simulation technique is whether the ill-conditioning of the problem (see fig.2 and 3) is due to the eigenfrequencies of the inclosed source surface. In fig. 8 the boundary error can be seen, calculated for the first resonance of the circle $kr = 2.4048255577$, as a function of the number of sources. It can be observed that the error is extremely large (see fig. 6), though remaining constant despite a substancial increase in the number of sources: 20 to 150 monopoles. In ill-conditioned problems the trend is toward na increase in the error na in the condition number when the number o sources increases (A. Bogomolny, 1985). Therefore, one can conclude that the ill-conditioning of the problem is a characteristic of the source simulation technique and is not caused by the eigenfrequencies of the internal space formed by the source surface.

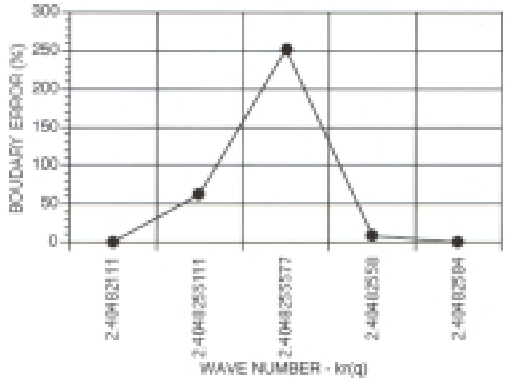



**Figure 6: Boundary error close to and at the resonance frequency of a circle** $kr(q) = 2.4048255577$

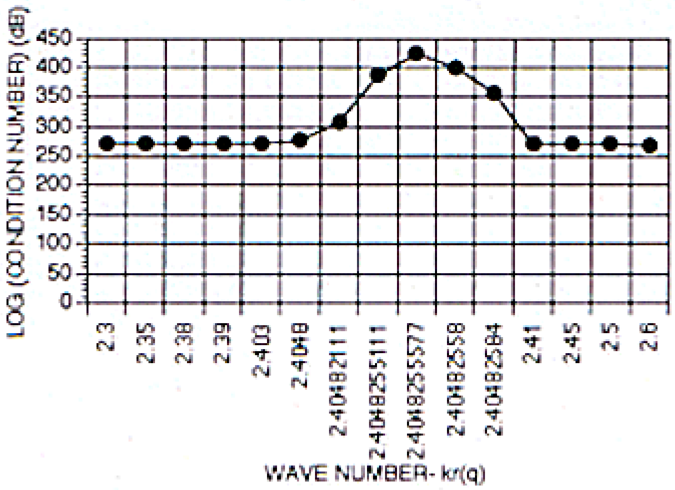



**Figure 7: Logarithm of the condition number at the proximities of the first resonance frequenz of a circle** $kr(q) = 2.4048255577$

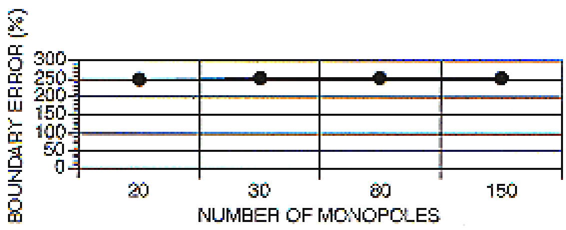



**Figure 8: Influence of the number of monopoles on the boundary error for** $kr(q) = 2.4048255577$

## 4. CONCLUDING REMARKS

The quality of the results obtained by the source simulation technique depends on the relationship between some parameters. The most relevant of them are the: shape of the inner source surface, the location of the sources at the source surface, and the number of sources. If one of these parameters is inadequately chosen, this will negatively influence the development of the whole numerical calculation.

In the case of the single-layer method, sources should not be positioned very close to the center of the body, as in this case the condition number of the matrix grows rapidily, meaning that it is becoming ill-conditioned. If the wave number is small, onde can position the sources close to the center of the body. The advantage of doing so is in the use of a small number of sources in order to minimize the boudary error, which is also translated into less computing time. On the other hand, as the wave number grows, the sources are located closer to the surface. However, the positioning of the sources should obbey a relationship between the smallest dimension of the source surface and the largest dimension of the body under study. For the case of a cylinder of radius $R$ and a source of radius $r(q)$, the relationship is given by $r(q)/R \leq 0.9$. Above this value the method becomes very unstable due to the occurence of singularities.

With respect to the shape of the source surface, number of the sources used there are two opposing effects. If the number of sources is too small the acoustic field cannot be reproduced with precision. If the number of sources is too large, both computing time and computational errors end up increasing. Numerical experiments led us to the conclusion that the ill-conditioning of the problem is not caused by the eigenfrequencies of the source surface, but it is a characteristic of the method itself. The method breaks down when the excitation frequency coincides with the eingenfrequencies of the inner space formed by the source surface. The numerical experiments have shown that the eigenfrequencies belong to a very narrow band. This way, for non-complex surfaces such as sphere or a circle, they can be easily calculated and avoided.

The greated disavantage in the use of the source simulation technique is in the fact that rules for the positioning of the source surface are not known a *priori*.The positioning of the source surface and in consequence of the sources themselves is based on the experience of the programmer. Further research is necessary to investigate how the method performs with complex surfaces. In that case the main question is about the shape of the source surface.

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